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# Galilei Invariance and Superfluidity II

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*Abstract*

Continuing the (heuristic) analysis of the mathematical structure of the Landau excitations, we find that in *one dimension* they may be described by a vector bundle over the base space of the boosts. The total space is a direct integral of all irreducible representations (of a given class) of the Galilei group. The existence of an energy-momentum spectrum requires the action of the boosts to be non-linear. This action can also be formulated as a superselection rule.

## *Formulation of the Problem*

In a previous article with the same title (Sen & Zahavi, 1972) it was pointed out that the Galilei transformation properties of $\mathbf{p}$ and $E$ in the low-lying (Landau) excitations in superfluid helium are identical with those of the eigenvalues of the infinitesimal space and time translation operators in the zero-mass representations (Levy-Leblond, 1963, 1971) of the Galilei group (Inönü & Wigner, 1952). As these representations had earlier been considered to be of dubious physical interest (Inönü & Wigner, 1952; Wightman, 1962), it was natural to ask whether, apart from the above-mentioned identity of the transformation properties, there existed a deeper relationship between these representations and the Landau excitations. The previous article implied that the answer was in the affirmative. There was, however, a gap in its argument, which was not appreciated by the authors at the time of its writing. In this article we point out this gap, examine its physical consequences and give a heuristic discussion of how it is to be filled. A detailed mathematical treatment will be presented at a later date.

The two problems which have to be analysed in this context are concerned respectively with localisability and the existence of an energy spectrum. The first has been adequately discussed in the previous article, and here we will examine only the consequences of the second. The energy and momentum of a Landau excitation are related in the rest frame by

$$E = E(p), \qquad p = |\mathbf{p}| \tag{1}$$

where the function $E(p)$ is not determined by the invariance group. In an inertial frame moving with velocity $\mathbf{v}$, the energy $E$ transforms to $E'$, given by

$$E' = E(p) + \mathbf{p}.\mathbf{v} \tag{2}$$

whereas the momentum remains unchanged:

$$\mathbf{p}' = \mathbf{p} \tag{3}$$

Consider the zero-mass zero-helicity irreducible representations $D_\lambda$ of the Galilei group. These are labelled by a positive number $\lambda$, $\lambda^2 = \mathbf{p}^2$ for any $\mathbf{p}$ occurring in $D_\lambda$. We work with $\delta$-normalisable vectors. The representation space $\mathscr{H}_\lambda$ of $D_\lambda$ is the linear space of vectors $|\mathbf{p}, E\rangle$ with complex coefficients and the scalar product

$$\langle \mathbf{p}, E | \mathbf{p}', E' \rangle = \delta(\mathbf{p} - \mathbf{p}')\,\delta(E - E') \tag{4}$$

and the ranges of $\mathbf{p}$ and $E$ in $\mathscr{H}_\lambda$ are respectively:

(i) $\mathbf{p} = R\mathbf{p}_0$, $\mathbf{p}_0$ a fixed vector with $\mathbf{p}_0{}^2 = \lambda^2$ and $R$ a three-dimensional rotation;

(ii) $-\infty < E < \infty$.

Since the allowed values of $E$ in $\mathscr{H}_\lambda$ fill the real line and $\mathscr{H}_\lambda$ is a linear vector space, *it is possible to superpose all real values of E for any* $\mathbf{p}$.

However, in the case of Landau excitations it is an observed fact that *the energy is fixed uniquely in the rest frame* (and therefore in other frames) *by the momentum* (equation (1)). Superpositions of different energies for the same momentum are never realised. We are, of course, disregarding the small but finite width $\Delta E$ of the excitation, due to its instability, at finite temperatures. It should, however, be remarked that this width is non-trivial, because there exists no *a priori* relationship between it and a momentum uncertainty $\Delta p$.

However, superposition of states with different momenta are *not* forbidden. In the first instance, the quantitative agreement with the specific heat formula at low temperatures indicates that it is correct to regard states with different $\mathbf{p}$ as different states of the same physical entity, the Landau excitation.† One would ordinarily expect to be able to superpose different states of the same quantum-mechanical entity. Next, Landau excitations are necessarily localised in a finite region, which implies that it is possible to superpose states with different $\mathbf{p}$ to achieve an approximation to a spatial $\delta$-function.

To sum up the discussion so far, it appears that for Landau excitations:

(i) The Galilei transformation properties of the energy $E$ and momentum $\mathbf{p}$ are the same as those of the eigenvalues of the infinitesimal time and space translation operators in a true representation of the Galilei group.

† The names 'phonon' and 'roton' refer to different momentum intervals in the spectrum of Landau excitations, just as 'violet' and 'red' does for electromagnetic radiation.

(ii) It is physically meaningful to superpose states with different $\mathbf{p}^2$.

(iii) It is *not* physically meaningful to superpose states with the same $\mathbf{p}$ but different $E$ (apart from the width $\Delta E$ mentioned earlier, which we will disregard for the present).

The restriction (iii) above is essential to give meaning to the concept of an energy-momentum spectrum. If there is no such restriction on the superposition principle, we would be left with a direct integral of true representations of the Galilei group. In such a representation the essential feature of the Landau excitations, namely the energy-momentum spectrum, would not be meaningful.

Thus we have to find a mathematical structure which contains (i), (ii) and (iii). Such a structure was described earlier as a 'reducible representation with a dispersion law $E = E(p)$'. However, a more precise characterisation, in terms of vector bundles, appears to exist. This will be developed in the following for the one-dimensional case. It appears that the three-dimensional case cannot be properly discussed except in a more rigorous mathematical treatment.

### *Vector Bundles*

*A fibre bundle* is a triple $\{K, \pi, M\}$ consisting of two topological spaces $K$ and $M$ and a continuous map or *projection* $\pi$ from $K$ onto $M$ (Hermann, 1966). $K$ is called the *total space* and $M$ the *base space*. For any $y \in M$, the set of $x \in K$ such that $\pi(x) = y$ is called the *fibre* at $y$. It is denoted by $\pi^{-1}(y)$. A fibre bundle is a *vector bundle* if every fibre is a vector space. That is, if $\pi(x) = y$ then $\pi(ax) = y$, and if $\pi(x) = \pi(x') = y$, then $\pi(ax + bx') = y$, where $x, x' \in K$, $a, b \in \mathbb{R}$ (or $\mathbb{C}$; accordingly we have real or complex vector bundles).

Observe that $ax$ and, for certain $x$ and $x'$, $ax + bx'$ must be meaningful (in $K$) in order to define a vector bundle. However, addition of elements from two *different* fibres is not generally defined. Therefore, in order to avoid absurdities, an element of $K$ should belong to one and only one fibre. This is ensured by the map $\pi$ being a projection.

Since addition is defined within a fibre but not between different fibres, a vector bundle can be visualised as a structure which is linear within a fibre but 'non-linear' along the base space.

Next, we shall devise a method for *restricting* a vector space to a vector bundle. Let $x_1 \dots x_n$ be reals, $x = (x_1 \dots x_n)$ a real $n$-tuple and $A$ the point set consisting of these $n$-tuples. Vector space structures can be imposed on $A$ in many different ways. Two of the relevant ones are via the following metrics:

$$d_n(x, x') = \left[ \sum_{i=1}^{n} (x_i - x_i')^2 \right]^{1/2} \tag{5}$$

$$\delta_n(x, x') = \delta(x_1 - x_1') \dots \delta(x_n - x_n') \tag{6}$$

where $\delta(x_i - x_i')$ is a Dirac delta. The pair $\{A, d_n(x,x')\}$ is $\mathbb{R}^n$, the $n$-dimensional Euclidean space. We will take the pair $\{A, \delta_n(x,x')\}$ to be $\mathscr{H}_\delta$, the (improper) *complex* Hilbert space spanned by the $\delta$-normalised vectors of real arguments in $\mathbb{R}^n$:

$$\langle x_1 \ldots x_n | x_1' \ldots x_n' \rangle = \delta_n(x, x') \tag{7}$$

The triple $\{A, d_n(x,x'), \delta_n(x,x')\}$ is a point-set with two different topologies defined on it. Now consider an $\mathbb{R}^m$: $\{B, d_m(y,y')\}$, $m < n$, and a projection $\pi(\mathbb{R}^n \to \mathbb{R}^m)$ which maps $\{A, d_n(x,x')\}$ onto $\{B, d_m(y,y')\}$. The fact that $\pi$ is a projection implies that if $x$ and $x'$ are two distinct points in $A$ which are mapped onto the same point $y$ in $B$, then every point of the form $ax + bx'$, $a, b \in \mathbb{R}$ is mapped onto the same $y$. This mapping induces a mapping $\mathscr{H}_\delta \to \mathbb{R}^m$ in an obvious manner. If

$$(x \in \mathbb{R}^n) \to (y \in \mathbb{R}^m)$$

then

$$(|x\rangle \in \mathscr{H}_\delta) \to (y \in \mathbb{R}^m)$$

Notice that $x, x' \to y$ implies $ax + bx' \to y$, and that these two in turn imply (i) $|x\rangle$, $|x'\rangle \to y$, (ii) $\alpha|x\rangle + \beta|x'\rangle \to y$, and (iii) $|ax + bx'\rangle \to y$, for all $a, b \in \mathbb{R}$ and all $\alpha, \beta \in \mathbb{C}$. Hence the fibre $\pi^{-1}(y)$ in $\mathscr{H}_\delta$ is the complex vector space spanned by the linearly independent vectors of the form

$$|\textstyle\sum a_\lambda x_\lambda\rangle$$

where $a_\lambda \in \mathbb{R}$ and $x_\lambda \to y$.

As one might expect, the above structure can also be reformulated in terms of a superselection rule in $\mathscr{H}_\delta$. The mapping $\pi$ partitions $\mathscr{H}_\delta$ into mutually disjoint fibres $\pi^{-1}(y)$. Since the fibres are sub-spaces of $\mathscr{H}_\delta$, for each $y \in M$ we can construct a projection operator $F(y)$:

$$F(y)^* = F(y), \qquad F(y)^2 = F(y)$$

such that

$$F(y)\,\mathscr{H}_\delta = \pi^{-1}(y)$$
$$F(y)\,F(y') = F(y')\,F(y) = 0, \qquad \text{for } y \neq y'$$

Actually, since $F(y)$ is a parametrised family of projection operators, its argument being a point $y \in M = \mathbb{R}^m$ (we assume that we have chosen a definite coordinate system in $\mathbb{R}^m$), it is more correct to write it in differential notation, $dF(y)$. In terms of the latter, we define the self-adjoint linear operator

$$\theta = \int_M y\, dF(y)$$

on $\mathscr{H}_\delta$. Suppose now that

(a) $\theta$ is an observable;
(b) $\theta$ commutes with *every* observable.

Then $\theta$ defines a continuous superselection rule on $\mathscr{H}_\delta$. It no longer makes sense to superpose elements in $\mathscr{H}_\delta$ belonging to different superselection sectors—i.e. to different fibres.

It should, perhaps, be stated explicitly that our fibre bundles are obtained from a vector space with an additional topological structure by a continuous mapping vis-a-vis this topological structure onto another topological space. Once this mapping is effected, the vector space structure is *discarded* except within each fibre. To emphasise this point, we will denote such bundles by $\{\pi, M\}$.

## *Landau Excitations in One Dimension*

In one dimension, rotations do not exist, $p$ and $v$ take values on the real line, an irreducible representation is denoted by a real number $p$, and equation (2) is replaced by

$$E' = E(p) + pv \tag{8}$$

where, if we disregard reflections, it is no longer necessary to take the absolute value of $p$ in $E(p)$. If $p > 0$, $E'$ is a monotonically increasing function of $v$. If $p < 0$, $E'$ decreases monotically with $p$. In either case, equation (8) has a unique solution in $v$ for given $p$, $E$ and $E'$, *unlike* equation (2).

Let $\mathscr{H}$ be the direct integral of these representations defined by the Lebesgue measure on the real $p$-line. We take $\mathscr{H}$ to be the total space. Next, we take for the base space the space of the boosts, i.e. the real line $-\infty < v < \infty$. Finally, we define the mapping $\pi: \mathscr{H} \to \mathbb{R}$ as follows:† $\pi^{-1}(v)$ is the vector space spanned by the vectors

$$|p, E(p) + pv\rangle, \qquad -\infty < p < \infty$$

It is immediately seen that:

(i) All $p$-values occur (and therefore localised states exist) in every fibre.

(ii) $\pi$ is a projection, i.e. two fibres $\pi^{-1}(v_1)$ and $\pi^{-1}(v_2)$ either have no element in common or else are identical.

(iii) Every fibre is invariant under space and time translations.

(iv) The boosts $G(u)$ act as follows on the fibres:

$$G(u)\,\pi^{-1}(v) = \pi^{-1}(u + v)$$

Hence the bundle $\{\pi, \mathbb{R}\}$ carries a realisation of the Galilei group which is linear in space and time translations but non-linear in the boosts. The notion of an energy spectrum is well defined in $\{\pi, \mathbb{R}\}$ and localisable states exist in every fibre. The bundle $\{\pi, \mathbb{R}\}$ gives a complete heuristic solution of the problem of the mathematical structure of Landau excitations in one dimension.

## *Concluding Remarks*

In three dimensions, the above heuristic method fails because equation (2) does not admit of a unique solution for $\mathbf{v}$ for given $E'$, $E(p)$ and $\mathbf{p}$. A more

† Observe that this is induced by the mapping $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, where $\mathbb{R} \times \mathbb{R}$ is the space of $(p, E)$ which is projected onto $\mathbb{R}$, the space of the boosts.

rigorous mathematical treatment is required before this problem can be satisfactorily resolved. In this context, it is worth pointing out that a state with a sharp value of the energy for a given momentum is *not* a normalisable state in the direct integral representation. It seems that just as the requirement of localisability forces one to abandon irreducibility, the requirement of normalisability will compel one to introduce a width, i.e. a finite lifetime for the Landau excitations.

A rigorous investigation of the mathematical structure of the Landau excitations appears to be interesting both from the physical and the mathematical viewpoints.

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